Short Papers

A New Approximation for the Capacitance of a Rectangular-Coaxial-Strip Transmission Line

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Abstract—**The method of conformal transformation is used to obtain the exact capacitance of a rectangular-coaxial-strip transmission line. An approximate form, which includes the edge-interaction capacitance of the strip, is obtained and is shown to reduce in an appropriate limit to a form obtained by other authors.**

Introduction

In deriving the capacitance of rectangular-coaxial cylinders whose inner conductor has zero thickness, the method of conformal transformation is frequently used to solve for the capacitance of a related problem where one of the side walls is taken to infinity [1]. To apply this result to the actual problem, a restriction must be placed on the ratio of the width of the inner conductor, $2w$, compared to the height of the outer conductor, $2b$, in order that interaction effects from the two edges of the inner conductor are not significant. However, the error resulting from such an *a priori* restriction cannot be assessed. In this short paper a more general approximation is obtained which reduces to the one given in [1] when the restriction on w/b, which follows naturally from this formalism, is made. In addition, the new approximation allows one to calculate the interaction effect for a limited range of w/b ratios. The result obtained in this short paper is directly applicable to the design of a matched TEM transmission cell currently being developed at the National Bureau of Standards for susceptibility and radiated emissions testing of electrically small devices [2].

Exact Capacitance of a Rectangular-Coaxial-Strip Transmission Line

A cross-sectional view of a rectangular-coaxial-strip transmission line is shown in Fig. 1 with an x–y coordinate system superimposed.

The strip of width $2w$ is located symmetrically inside a shielded enclosure of height $2b$ and width $2a$ and is assumed to have negligible thickness. In addition, the strip is located a distance g from each vertical side wall and is embedded in a homogeneous dielectric of permittivity ε_0. The reason for choosing an unsymmetrically located coordinate system is to facilitate obtaining an approximate formula for the capacitance.

To determine the capacitance, the method of conformal transformation will be used, whereby the structure in Fig. 1 is transformed into a simpler configuration whose capacitance is already known. Since it is well known that capacitance is invariant under a conformal transformation, the formula obtained will also be applicable to the shielded stripline configuration of Fig. 1.

Since we have symmetry with respect to the x axis, we will calculate the capacitance between the upper plate, A–F–E–D and the stripline, B–C. The total capacitance is then twice this figure, since we have effectively two capacitors in parallel. The region A–D–E–F may be mapped into the upper half of a complex t plane via the Schwarz–Christoffel transformation [3] which, due to symmetry, can be expressed in terms of Jacobian elliptic functions [4, pp. 7–15]. The transformation is given by [4, p. 58]

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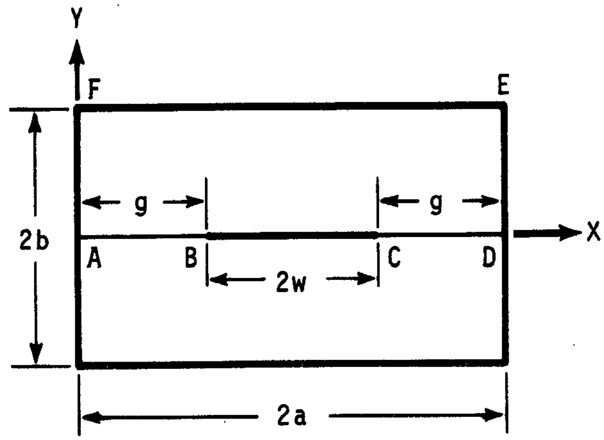

Fig. 1. Cross section of a rectangular-coaxial-strip transmission line.

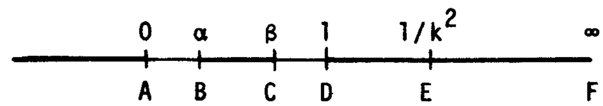

Fig. 2. Complex t plane.

$$mz = \int_0^t \frac{dt'}{[4t'(1-t')(1-k^2t')]^{1/2}} \tag{1}$$

or alternatively by

$$t = \text{sn}^2\,(mz,k) \tag{2}$$

where sn is a Jacobian elliptic function of modulus k

$$m = \frac{K(k')}{b} \tag{3}$$

and

$$z = x + iy. \tag{4}$$

Here $K(k)$ and $K(k')$ are complete elliptic integrals of the first kind of moduli k and k', respectively [4, pp. 16–25], and

$$k' = [1 - k^2]^{1/2}. \tag{5}$$

The modulus k can be determined from the requirement that

$$\frac{K(k)}{K(k')} = \frac{2a}{b}. \tag{6}$$

(See, for example [5], where the value of k^2 is tabulated for a given ratio, $K(k')/K(k)$.) Under the transformation given by (2), the region A–D–E–F in the z plane is mapped into the upper half of the t plane as shown in Fig. 2. Using (2) and elliptic function identities, α and β can be calculated as

$$\alpha = \text{sn}^2\, mg = \text{sn}^2\, \xi \tag{7}$$

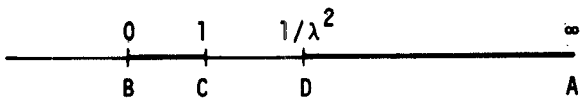

Fig. 3. Complex u plane.

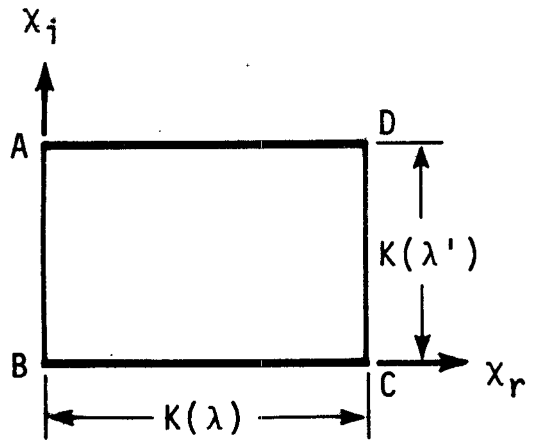

Fig. 4. Complex χ plane.

and

$$\beta = \text{sn}^2\, m(2a - g) = \text{cn}^2\, \xi/\text{dn}^2\, \xi \tag{8}$$

where

$$\xi \equiv mg \tag{9}$$

and cn and dn are also Jacobian elliptic functions each of which has modulus k. For convenience, we now make an intermediate transformation from the t plane to a complex u plane defined by

$$u = \frac{\beta}{t}\left(\frac{t - \alpha}{\beta - \alpha}\right). \tag{10}$$

The u plane is shown in Fig. 3. λ in Fig. 3 can be found using (10) and substituting $t = 1$. Thus

$$\frac{1}{\lambda^2} = \beta\left(\frac{1 - \alpha}{\beta - \alpha}\right) \tag{11}$$

and substituting for α and β from (7) and (8), we obtain

$$\lambda^2 = \frac{\text{cn}^2\, \xi - \text{sn}^2\, \xi\, \text{dn}^2\, \xi}{\text{cn}^2\, \xi[1 - \text{sn}^2\, \xi]}. \tag{12}$$

Using elliptic function identities, (12) reduces to

$$\lambda^2 = 1 - k'^2\left(\frac{\text{sn}\, \xi}{\text{cn}\, \xi}\right)^4 \tag{13}$$

and defining a complementary modulus, λ', as

$$\lambda' = [1 - \lambda^2]^{1/2} \tag{14}$$

we have from (13) and (14)

$$\lambda' = k'\left(\frac{\text{sn}\, \xi}{\text{cn}\, \xi}\right)^2. \tag{15}$$

In the final transformation, we map the upper half of the u plane into a rectangular region in a complex χ plane. The transformation is given by

$$u = \text{sn}^2\, (\chi,\lambda) \tag{16}$$

and the χ plane is shown in Fig. 4. From Fig. 4, it is evident that the capacitance is just given by the ordinary parallel-plate capacitor formula, that is

$$\frac{C}{\varepsilon_0 L} = \frac{K(\lambda)}{K(\lambda')}. \tag{17}$$

Therefore, the total capacitance C_0 of the rectangular-coaxial-strip transmission line per unit length L is just twice that given by (17).

$$\frac{C_0}{\varepsilon_0 L} = 2\,\frac{K(\lambda)}{K(\lambda')}. \tag{18}$$

Approximate Expression for the Capacitance

Since (6) and (18) both involve ratios of complete elliptic integrals, the following approximation is particularly useful [6]:

$$\frac{K(\delta)}{K(\delta')} \simeq \frac{1}{\pi}\ln\left(2\,\frac{1 + \sqrt{\delta}}{1 - \sqrt{\delta}}\right), \qquad (\delta^2 > \tfrac{1}{2}). \tag{19}$$

Using (19) we can write approximate expressions for (6) and (18), respectively, as follows:

$$\frac{2a}{b} \simeq \frac{1}{\pi}\ln\left(2\,\frac{1 + \sqrt{k}}{1 - \sqrt{k}}\right), \qquad (k^2 > \tfrac{1}{2}) \tag{20}$$

$$\frac{C_0}{\varepsilon_0 L} \simeq \frac{2}{\pi}\ln\left(2\,\frac{1 + \sqrt{\lambda}}{1 - \sqrt{\lambda}}\right), \qquad (\lambda^2 > \tfrac{1}{2}). \tag{21}$$

Equations (20) and (21) may be written alternatively as

$$\frac{4a}{b} \simeq \frac{2}{\pi}\ln\,[2(1 + \sqrt{k})^2(1 + k)] - \frac{2}{\pi}\ln\,(1 - k^2) \tag{22}$$

$$\frac{C_0}{\varepsilon_0 L} \simeq \frac{2}{\pi}\ln\,[2(1 + \sqrt{\lambda})^2(1 + \lambda)] - \frac{2}{\pi}\ln\,(1 - \lambda^2). \tag{23}$$

Subtracting (22) from (23) we obtain

$$\frac{C_0}{\varepsilon_0 L} - \frac{4a}{b} \simeq \frac{2}{\pi}\ln\left(\frac{k'^2}{\lambda'^2}\right) + \frac{2}{\pi}\ln\left\{\left(\frac{1 + \sqrt{\lambda}}{1 + \sqrt{k}}\right)^2\left(\frac{1 + \lambda}{1 + k}\right)\right\} \tag{24}$$

and substituting from (15)

$$\frac{C_0}{\varepsilon_0 L} \simeq 4\left\{\frac{a}{b} + \frac{2}{\pi}\ln\left(\frac{\text{cn}\, \xi}{\text{sn}\, \xi}\right)\right\} + \frac{2}{\pi}\ln\left\{\left(\frac{1 + \sqrt{\lambda}}{1 + \sqrt{k}}\right)^2\left(\frac{1 + \lambda}{1 + k}\right)\right\}. \tag{25}$$

In (20) the restriction that $k^2 > 1/2$ is equivalent to requiring $(b/2a) < 1$, since when $k^2 = 1/2$, $K(k') = K(k)$. If we make the somewhat more stringent requirement that $(b/a) < 1$, which is equivalent to $k^2 > 0.97$, then (25) may be further simplified by noting that for $k \simeq 1$, $\text{cn}\, \xi \simeq \text{sech}\, \xi$; $\text{sn}\, \xi \simeq \tanh \xi$; and $\xi \simeq (\pi g/2b)$. Thus

$$\frac{C_0}{\varepsilon_0 L} \simeq 4\left\{\frac{a}{b} - \frac{2}{\pi}\ln\left(\sinh\frac{\pi g}{2b}\right)\right\} - \frac{\Delta C}{\varepsilon_0 L} \tag{26}$$

where

$$\frac{\Delta C}{\varepsilon_0 L} = \frac{2}{\pi}\ln\left\{\left(\frac{1 + \sqrt{k}}{1 + \sqrt{\lambda}}\right)^2\left(\frac{1 + k}{1 + \lambda}\right)\right\}. \tag{27}$$

An alternative form of (26) may be obtained by using the following identity:

$$\sinh\left(\frac{\pi g}{2b}\right) = \frac{e^{(\pi g/2b)}}{\left[1 + \coth\left(\frac{\pi g}{2b}\right)\right]} \tag{28}$$

with the result

$$\frac{C_0}{\varepsilon_0 L} \simeq 4\left\{\frac{w}{b} + \frac{2}{\pi}\ln\left(1 + \coth\frac{\pi g}{2b}\right)\right\} - \frac{\Delta C}{\varepsilon_0 L}. \tag{29}$$

In this form it is easy to identify the first term in (29) as the plate capacitance between the stripline and the horizontal walls, and the second term as the fringing capacitance between the edges of the stripline and the side walls. For large gaps, the fringing term approaches $(8/\pi)\ln 2$, as expected [7, p. 515].

It is interesting to note that the first term on the right-hand side of (29) is the same formula given by Chen [7] and originally derived by Cohn [1]. Cohn's formula was derived assuming that the width of the center septum, $2w$, was very large compared to the plate separation, $2b$. This is equivalent to assuming that the two edges of the septum do not interact. ΔC, then, in (29) can be interpreted as a correction term needed to account for the interaction between the two edges. From (27) it can be seen that ΔC will be negligibly small if λ is near one (or λ' is near zero) since k is near one. From (15) λ'^2 is given approximately by

$$\lambda'^2 \simeq k'^2 \sinh^4\left(\frac{\pi g}{2b}\right). \tag{30}$$

It can be seen from (30) that for small gaps, λ' is always much less than one. For large gaps, it can be shown that (30) further reduces to

$$\lambda'^2 \simeq e^{-2\pi(w/b)} \tag{31}$$

by using the approximate expression for the modulus, k, given by Anderson [8]. From (31) it can easily be verified that λ' will be negligibly small, and hence ΔC may be neglected if

$$\frac{w}{b} \geq \frac{1}{2}. \tag{32}$$

In (21) we have the restriction that $\lambda^2 > 1/2$, or equivalently $\lambda'^2 < 1/2$. From (31) it can be seen that $\lambda'^2 < 1/2$ if

$$\frac{w}{b} \geq \frac{1}{2\pi}\ln 2 \simeq 0.1. \tag{33}$$

So for the range: $1/10 < w/b < 1/2$, ΔC is not negligible and must be calculated using (27), (31), and $k \simeq 1$.

The approximate formula for the capacitance given in (26) is plotted in Fig. 5 with a dashed line for $\Delta C = 0$. The exact formula using (6), (15), and (18) is plotted using a solid line. The two curves agree almost identically except where $w/b < 1/2$. This discrepancy can be attributed, however, to the ΔC term which was neglected.

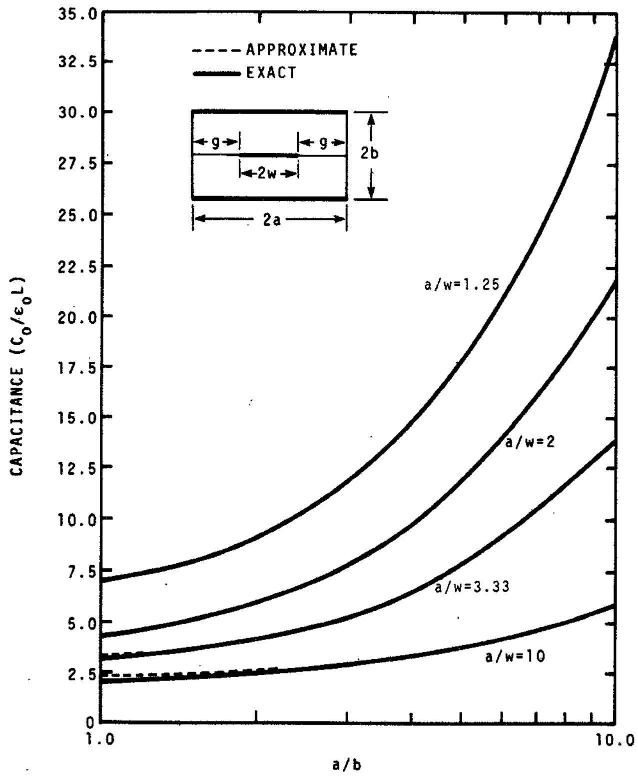

Fig. 5. Capacitance of a rectangular-coaxial-strip transmission line.

Conclusions

The exact and an approximate form for the capacitance of the rectangular-coaxial-strip transmission line have been presented. The approximate form enables one to evaluate the edge-interaction capacitance for a limited range of w/b ratios. For w/b ratios greater than 1/2, our approximation was shown to reduce to those obtained by other authors who neglected the edge-interaction capacitance. Thus we have found the restrictions that must be observed when using their approximation.

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A Coplanar Waveguide with Thick Metal-Coating

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***Abstract*—A theoretical method is presented for the analysis of a coplanar waveguide with thick metal-coating. Numerical results are given and compared with published data. It is shown that the metal-coating thickness of the coplanar waveguide causes an increase in wavelength and a decrease in characteristic impedance and that the changes are about the same as those of a slot line.**

Introduction

A coplanar waveguide (CPW) has been investigated on the basis of a quasi-static approximation [1], [2], and recently Knorr and Kuchler [3] obtained the frequency dependence of

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